

# Experimental quantum communication complexity

Pavel Trojek<sup>1,2</sup>, Christian Schmid<sup>1,2</sup>, Mohamed Bourennane<sup>1,2</sup>,

Časlav Brukner<sup>3</sup>, Marek Żukowski<sup>4</sup>, and Harald Weinfurter<sup>1,2</sup>

<sup>1</sup> *Max-Planck-Institut für Quantenoptik, D-85748 Garching, Germany*

<sup>2</sup> *Sektion Physik, Ludwig-Maximilians-Universität, D-80799 München, Germany*

<sup>3</sup> *Institut für Experimentalphysik, Universität Wien, Boltzmanngasse 5, A-1090, Wien, Austria*

<sup>4</sup> *Instytut Fizyki Teoretycznej i Astrofizyki Uniwersytet Gdański, PL-80-952 Gdańsk, Poland*

(Dated: June 23, 2004)

We prove that the fidelity of two exemplary communication complexity protocols, allowing for an  $N-1$  bit communication, can be exponentially improved by  $N-1$  (unentangled) qubit communication. Taking into account, for a fair comparison, all inefficiencies of state-of-the-art set-up, the experimental implementation outperforms the best classical protocol, making it the candidate for multi-party quantum communication applications.

PACS numbers: 03.67.Hk, 42.65.Lm

Quantum information science transgresses limitations of conventional information transfer, cryptography and computation. Recently, significant advantages were recognized when applying quantum phenomena in the field of communication complexity problems (CCP's) [1]. There, separated parties, performing *local* computations, exchange information in order to accomplish some *globally* defined task. Two types of CCP's are distinguished: the first minimizes the amount of information exchange necessary to solve the task with certainty [2, 3, 4]. The second maximizes the probability of successfully solving the task for restricted amount of information [4, 5, 6]. Such studies aim, e.g., at a speed up of distributed computations by increasing the communication efficiency, or at an optimization of VLSI circuits and data structures [7].

Quantum CCP protocols, using multi-particle entanglement, were proven to be clearly superior with respect to classical ones [2, 3, 4, 5, 6]. However, the technology of entanglement based multi-party quantum communication is still in a premature stage. A recent reformulation of quantum CCP's pointed out that even the communication employing single qubits may outperform classical CCP's [8, 9, 10]. Such a simplification would be of tremendous importance, as it would make a multi-party communication task technologically comparable to quantum key distribution, the only commercial application of quantum information science so far.

Here we prove that, for CCP's with restricted communication, the superiority of the single qubit assisted protocols over the corresponding classical ones may increase even exponentially with the number of partners. Furthermore, using parametric down-conversion as a source of heralded single qubits, we experimentally show that quantum protocols solve two exemplary CCP's more efficiently, even with the limited detection efficiency inherent in real single-photon experiments. By solving these CCP's with a sequential transfer of a single qubit only, we demonstrate a generic way of bringing multi-party quantum communication schemes much closer to realistic applications.

Let us first introduce the two CCP's analyzed and implemented here. The first one, problem A, is the so called *modulo-4 sum* problem [3, 4, 10]. Imagine  $N$  separated partners  $P_1, \dots, P_N$ . Each of them receives a two-bit string  $X_k$ , ( $X_k = 0, 1, 2, 3; k = 1, \dots, N$ ). The  $X_k$ 's are distributed such that their sum is even, i.e.  $(\sum_{k=1}^N X_k) \bmod 2 = 0$ . No partner has any information whatsoever on the values received by the others. The partners then communicate with the common goal that one of them, say  $P_N$ , can tell whether the sum modulo-4 of all input strings is equal 0 or 2. That is,  $P_N$  announces the value of the dichotomic, i.e. equal  $\pm 1$ , function  $T(X_1, \dots, X_N)$  given by  $T_A(X_1, \dots, X_N) = 1 - (\sum_{k=1}^N X_k) \bmod 4$  (for an alternative formulation see footnote [11]). The partners can freely choose how to communicate information about their  $X_k$ , i.e. they can choose between sequential communication from one to the other or any arbitrary tree-like structure ending at the last party  $P_N$ . However, the total amount of communication is restricted to only  $N - 1$  bits (classical scenario).

Problem B has a similar structure, but now  $N$  *real* numbers  $X_1, \dots, X_N \in [0, 2\pi)$  with probability density

$$p_B(X_1, \dots, X_N) = \frac{1}{4(2\pi)^{N-1}} |\cos(X_1 + \dots + X_N)| \quad (1)$$

are distributed to the partners. Their task is to compute whether  $\cos(X_1 + \dots + X_N)$  is positive or negative, i.e. to give the value of the function  $T_B = S[\cos(\sum_{k=1}^N X_k)]$ , where  $S(x) = x/|x|$ . The communication restriction is the same as for problem A.

To find the best performing classical protocols for these CCP's, we first rewrite the random inputs  $X_k$ . For the task A we put  $X_k = (1 - y_k) + x_k$ , where  $y_k \in \{-1, 1\}$ ,  $x_k \in \{0, 1\}$ . For the task B we write  $X_k = \pi(1 - y_k)/2 + x_k$ , with  $y_k \in \{-1, 1\}$ ,  $x_k \in [0, \pi)$ . Note that the dichotomic variables  $y_k$  are not restricted by the probability distributions for the  $X_k$ 's. Thus they are completely random. The global task function  $T$  can now be put as  $T = \prod_{k=1}^N y_k f(x_1, \dots, x_N)$ , see [12], and  $p(X_1, \dots, X_N) = 2^{-N} p'(x_1, \dots, x_N)$ .

Depending on the value of the product of all  $y_k$ 's the value of  $T$  flips between  $\pm 1$ . Thus, if information on  $y_k$  of any of the partners is omitted in the course of the protocol, the result is completely random. This implies that all  $N$  partners must be involved in an unbroken communication structure. Because of the restriction to maximum  $N - 1$  bits of communication, each of the partners must send only *one* bit, except for the last one,  $P_N$ , who gives the result [13].

Each of these one-bit messages encodes the value of a dichotomic function  $e_k = \pm 1$ . It depends on the local input number  $X_k$  and possibly on information  $\{e_l, e_m, \dots\}$  already received from other partners. Due to the highly restricted form of two valued functions, see [14], one can express any  $e_k$  in the form  $e_k = b_k(x_k, e_l, e_m, \dots) + c_k(x_k, e_l, e_m, \dots)y_k$ . In order to obtain a non-random final result,  $e_k$  must depend on  $y_k$ . Thus,  $b_k$  must be equal 0, while  $c_k$  itself is now a dichotomic function. Continuing the expansion of  $c_k(x_k, e_l, e_m, \dots)$  and keeping in mind that all previous messages received by the  $k$ -th partner must be taken into account, we obtain  $e_k = y_k a_k(x_k) e_l e_m \dots$ , where  $a_k$  is again a dichotomic function depending only on the local input  $x_k$ . Next, one expands in a similar way  $e_l, e_m, \dots$ , which leads to  $e_k = y_k a_k(x_k) y_l a_l(x_l) y_m a_m(x_m) \dots$ . The final answer,  $e_N$ , given by  $P_N$  must have the same structure as  $e_k$  and therefore, it must be equal to  $e_N = \prod_{i=1}^N a_i(x_i) y_i$ , [15].

If the answer is correct, then  $T = e_N$ , and thus  $T \cdot e_N = 1$ . Otherwise,  $T \cdot e_N = -1$ . Thus, one can introduce the measure  $F$  (fidelity) of the average success in the form

$$F_c = \left| \sum_{X_1, \dots, X_N} p(X_1, \dots, X_N) T(X_1, \dots, X_N) e_N \right|. \quad (2)$$

For the problem B the summations are replaced by integrations. The probability of success,  $P$ , is given by  $P = (1 + F)/2$ . For the best classical protocols of the CCP's given above we obtain

$$\begin{aligned} F_c &= \left| 2^{-N} \sum_{x_1, \dots, x_N} \sum_{y_1, \dots, y_N = \pm 1} p'(x_1, \dots, x_N) \prod_{l=1}^N y_l \right. \\ &\quad \times f(x_1, \dots, x_N) e_N(X_1, \dots, X_N) \left. \right| \\ &= \left| \sum_{x_1, \dots, x_N} g(x_1, \dots, x_N) a_1(x_1) \dots a_N(x_N) \right|, \quad (3) \end{aligned}$$

where we denoted the product  $p'(x_1, \dots, x_N) f(x_1, \dots, x_N)$  by  $g(x_1, \dots, x_N)$ . Since  $F_c$  depends on the product of local functions  $|a_i(x_i)| \leq 1$ , it is bounded from above, i.e.,  $F_c \leq B(N)$  [6].

The bounds  $B(N)$  for our problems A and B can be easily calculated. In both cases the fidelity decreases exponentially with number  $N$  of parties. For task A one has  $F_{c,A} = 2^{-K+1}$ , where  $K = N/2$  and  $K = (N+1)/2$  for even and odd number of parties, respectively. This *analytic* result, valid for arbitrary  $N$ , confirms the numerical simulations of [10] for small  $N$ . For task B we

derived  $F_{c,B} = (2/\pi)^{N-1}$ . Due to the formal analogies the integrals needed to get this result already appeared in the derivation of a Bell inequality involving continuous range of settings [16].

The Holevo bound [17] limits the information storage capacity of a qubit to exactly one classical bit. Thus, we restrict the maximum communication exchange for quantum protocols of the presented CCP's to  $N - 1$  qubits, or alternatively, to  $N - 1$ -times exchange of a *single* qubit. The solution of task A starts with a qubit in the state  $|\psi_i\rangle = 2^{-1/2}(|0\rangle + |1\rangle)$ . Parties sequentially act on the qubit with the unitary phase-shift transformation of the form  $|0\rangle\langle 0| + e^{i\pi/2 X_k} |1\rangle\langle 1|$ , in accordance with their local data. After all  $N$  phase shifts the state is

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/2(\sum_{k=1}^N X_k)} |1\rangle). \quad (4)$$

Since the sum over  $X_k$  is even, the phase factor  $e^{i\pi/2(\sum_{k=1}^N X_k)}$  is equal to the dichotomic function  $T_A$  to be computed. Therefore, a measurement of the qubit in the basis  $(|0\rangle \pm |1\rangle)/\sqrt{2}$  reveals the value of  $T_A$  with fidelity  $F_{q,A} = 1$ , that is, always correctly.

Our quantum protocol for task B starts with a qubit in the same state  $|\psi_i\rangle$ . Each party performs according to his/her local data a unitary transformation  $|0\rangle\langle 0| + e^{iX_k} |1\rangle\langle 1|$ . Thus, the final state is

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\sum_{k=1}^N X_k} |1\rangle). \quad (5)$$

The last party makes the same measurement as in task A. The probability for the detection of state  $2^{-1/2}(|0\rangle \pm |1\rangle)$ , which we associate with the result  $r = \pm 1$ , is given by  $P(\pm) = [1 \pm \cos(\sum_{k=1}^N X_k)]/2$ . The expectation value for the final answer  $e_N = r$  is  $E(X_1, \dots, X_N) = P(+)-P(-)$ , and reads  $\cos(\sum_{k=1}^N X_k)$ . The fidelity of  $e_N$ , with respect to  $T_B$  is

$$\begin{aligned} F_{q,B} &= \int_0^{2\pi} dX_1 \dots \int_0^{2\pi} dX_N p_B(X_1, \dots, X_N) \\ &\quad \times T_B(X_1, \dots, X_N) E(X_1, \dots, X_N). \quad (6) \end{aligned}$$

With the actual forms of  $p_B$ ,  $T_B$ , and  $E$ , one gets  $F_{q,B} = \pi/4$ , i.e., the protocol gives the correct value of  $T_B$  with probability  $P_{q,B} = (1 + \pi/4)/2 \approx 0.892$ .

For both problems the classical fidelity  $F_c$  or the probability of success  $P_c$  decreases exponentially with number  $N$  of parties to the value corresponding to a random guess of the result of  $T$ , i.e. to the value achievable without any communication at all. In contrast,  $P_q$  remains constant for any  $N$  and reaches 1 for task A, and  $\approx 0.892$  for task B. The simple, one qubit assisted quantum protocol clearly outperforms the best classical protocols without any shared multi-particle entanglement (!), utilizing only the coherence properties of the transmitted qubit.

We implemented the quantum protocols for  $N = 5$  parties, using a heralded single photon as the carrier of the qubit communicated from one partner to the other

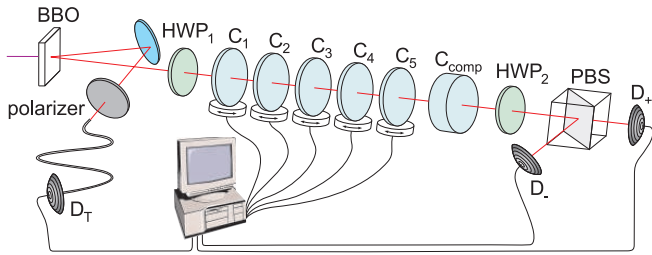


FIG. 1: Set-up for quantum CCP. Pairs of orthogonally polarized photons are emitted from a BBO crystal via the type-II SPDC process. The detection of one photon as trigger at  $D_T$  indicates the existence of the other one used in protocol. The polarization state is prepared with a half-wave plate ( $HWP_1$ ) and a polarizer, placed in the trigger arm. Each of the five parties introduces a phase-shift by the rotation of a birefringent  $YVO_4$  crystal ( $C_1$  to  $C_5$ ). The last party performs the analysis of a photon-polarization state using a half-wave plate ( $HWP_2$ ) followed by a polarizing beam-splitter (PBS).

[18]. The qubit was encoded in polarization. The computational basis, “0” and “1”, corresponds to horizontal  $H$  and vertical  $V$  linear polarization, respectively. The data  $X_k$  of each party was encoded on the qubit via a phase shift using birefringent materials. The last party performed a measurement in the  $2^{-1/2}(|H\rangle \pm |V\rangle)$  basis in order to obtain the final value  $T$ .

The schematic set-up is shown in Fig. 1. Photon pairs are produced via spontaneous parametric down-conversion (SPDC). The detection of one photon by detector  $D_T$  as a trigger heralds the existence of the other one used in protocol. The narrow gate window of 4 ns for observing the coincidence detection between these two photons along with the single-count rates of  $\sim 140000 \text{ s}^{-1}$  at the detectors  $D_+$  and  $D_-$  warrant that the recorded data are due to single photons only. Type-II SPDC in 2 mm thick  $\beta$ -barium borate (BBO) crystal, pumped by a single-mode laser diode (402.5 nm, 10 mW) is used, emitting pairs of orthogonally polarized photons at  $\lambda = 805 \text{ nm}$  ( $\Delta\lambda \approx 6 \text{ nm}$ ). Filtering of the vertical polarization of trigger photons by a polarizer, ensures that the protocol photon has horizontal polarization initially. A half-wave plate ( $HWP_1$ ) transforms the state of the photon to  $2^{-1/2}(|H\rangle + |V\rangle)$  as required in protocol.

For a fair comparison between the classical protocol and the quantum protocol, no heralded events are discarded, even if the detection of the protocol photon fails. In such a case it is still allowed to guess the value of  $T$ . This works with probability  $1/2$ , and leads to very demanding experimental requirements for unambiguous demonstration of the enhanced efficiency of qubit-assisted CCP compared to its classical counterpart [10]. In particular, high detection efficiency of the heralded photons, i.e. high coincidence/single ratio for our set-up, is essential.

In order to minimize the events where no photon was detected, the yield of heralded photons was maximized by adopting an unbalanced SPDC scheme. That means,

we select a restricted spatial mode with well defined polarization of the trigger photons by coupling them into single-mode fibre behind a polarizer, whereas no spatial filtering is performed on the protocol photons. With such configuration we observed  $\approx 5000$  trigger events per second with  $\approx 2400$  coincident events per second of protocol detections, i.e. an overall detection efficiency of  $\approx 0.48$ , close to the limit given by the detector efficiency of the avalanche photodiodes used, which was about 55% for our operating wavelength.

The individual phase shifts of parties are implemented by rotating 200  $\mu\text{m}$  thick Yttrium-Vanadate ( $YVO_4$ ) birefringent crystals ( $C_i$ ) along their optic axis, oriented perpendicularly to the beam. An additional  $YVO_4$  crystal ( $C_{\text{comp}}$ ) compensates dispersion effects. To analyze the polarization state of photons in the desired basis, a half wave-plate ( $HWP_2$ ) followed by polarizing beam-splitter (PBS) is used.

The protocols were run many times, to obtain sufficient statistics. Each run took about one second. It consisted of generating a set of pseudorandom numbers obeying the specific distribution, subsequent setting of the corresponding phase shifts by the rotations of  $YVO_4$  crystals, and opening detectors for a collection time window  $\tau$ . The limitation of communicating one qubit per run requires that only these runs, in which exactly one trigger photon is detected during  $\tau$ , are selected for the evaluation of the probability of success  $P_{\text{exp}}$ . To maximize the number of such runs,  $n$ , the length of  $\tau$  was optimized to 200  $\mu\text{s}$  assuming Poissonian photon-number distribution of SPDC photons.

In order to determine the probability of success from the data acquired during the runs we have to distinguish the following two cases. First, the heralded photon is detected, which happens with probability  $\eta$  given by the coincidence/single ratio. Then, the answer on the value of the function  $T$  can be based on the measurement result. However, due to experimental imperfections in the preparation of the initial state, the setting of the desired phase shifts and the polarization analysis, the answer is correct only with probability  $\gamma$ , which must be compared with the theoretical limits given by  $P_{q,A}$  and  $P_{q,B}$  for the task A and B, respectively. Second, with the probability  $1 - \eta$  the detection of the heralded photon fails. Forced to make a random guess, one gives the correct answer in half of the cases. This leads to an overall success probability  $P_{\text{exp}} = \eta\gamma + (1 - \eta)0.5$ , or a fidelity of  $F_{\text{exp}} = \eta(2\gamma - 1)$ .

Due to a finite measurement sample, our experimental results for the success probability are distributed around the value  $P_{\text{exp}}$  as shown in Fig. 2 for both tasks. The width of the distribution is interpreted as the error in the experimental success probability. For task A we obtain a quantum success probability of  $P_{\text{exp},A} = 0.711 \pm 0.005$ . The bound  $P_{c,A} = 5/8$  for the optimum classical protocol is violated by 17 standard deviations. For the task B we reached  $P_{\text{exp},B} = 0.669 \pm 0.003$ , whereas the classical bound is  $P_{c,B} \approx 0.582$ . The violation is by 29 standard deviations, [19]. Table I summarizes the relevant experi-

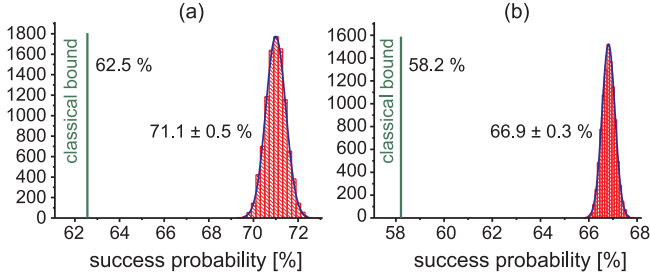


FIG. 2: Histograms of measured quantum success probabilities (a) for the task A and (b) for the task B. The bounds for optimum classical protocols are displayed as well.

TABLE I: Experimental parameters

	$n$	$\eta$	$\gamma$
task A	6692	$0.452 \pm 0.010$	$0.966 \pm 0.003$
task B	18169	$0.471 \pm 0.006$	$0.858 \pm 0.004$

mental parameters  $n$ ,  $\eta$  and  $\gamma$  for both tasks.

In conclusion, we have proven and experimentally demonstrated the superiority of quantum communication over its classical counterpart for distributed computational tasks by solving two exemplary CCP's. For

nontrivial CCP's, where the input from all the partners is required in order to obtain a non-random final result, the best classical fidelity goes exponentially to 0 with increasing number  $N$  of partners. Yet, the fidelity stays constant and independent on  $N$  for our single qubit assisted protocols.

In our experimental realization we have reached higher-than-classical performance even when including all experimental imperfections of state-of-the-art technologies. Thus, by successfully performing fair and real comparison with the classical scenario with present-day technology we clearly illustrate the potential of the implemented scheme in real applications of multi-party quantum communication. Most importantly, our method gives a generic prescription to simplify multi-party quantum communication protocols. For example, multi-party secret-sharing protocols employing multi-qubit GHZ-states and local operations only, can now be directly transformed to single-qubit applications, thereby significantly enhancing their applicability [20].

M.Ž. was supported by Profesorial Subsidy of FNP, and by MNIi grant No PBZ-MIN-008/ P03/ 2003. This work was supported by the DFG, EU-FET (RamboQ, IST-2001-38864), Marie-Curie program and DAAD/KBN exchange program.

- 
- [1] A. C.-C. Yao, in *Proceedings of the 11th Annual ACM Symposium on Theory of Computing*, (ACM Press, NewYork, 1979), p. 209.
  - [2] R. Cleve and H. Buhrman, *Phys. Rev. A* **56**, 1201 (1997).
  - [3] H. Buhrman, W. van Dam, P. Høyer, and A. Tapp, *Phys. Rev. A* **60**, 2737 (1999).
  - [4] H. Buhrman, R. Cleve, and W. van Dam, e-print quant-ph/9705033.
  - [5] L. Hardy and W. van Dam, *Phys. Rev. A* **59**, 2635 (1999).
  - [6] Č. Brukner, M. Żukowski, and A. Zeilinger, *Phys. Rev. Lett.* **89**, 197901 (2002); Č. Brukner, M. Żukowski, J.-W. Pan, and A. Zeilinger, *Phys. Rev. Lett.* **92**, 127901 (2004).
  - [7] E. Kushilevitz and N. Nisan, *Communication complexity* (Cambridge University Press, England, 1997).
  - [8] Harry Buhrman, Richard Cleve, and Avi Wigderson, in *Proceedings of the 30th Annual ACM Symposium on Theory of Computing*, (ACM Press, New York, 1998), p. 63.
  - [9] R. Raz, in *Proceedings of the 31st Annual ACM Symposium on Theory of Computing*, (ACM Press, New York, 1999), p. 358.
  - [10] E. F. Galvão, *Phys. Rev. A* **65**, 012318 (2001).
  - [11] An alternative description, simplifying calculations and making the connection with task B more visible, puts the probability distribution for local data as  $p_A(X_1, \dots, X_N) = 2^{-2N+1} |\cos(\frac{\pi}{2} \sum_{k=1}^N X_k)|$  and the global task function as  $T_A(X_1, \dots, X_N) = \cos(\frac{\pi}{2} \sum_{k=1}^N X_k)$ .
  - [12] Consequently,  $f = f_A = \cos(\frac{\pi}{2} \sum_{k=1}^N x_k)$ , and  $f = f_B = S[\cos(\sum_{k=1}^N x_k)]$ .
  - [13] Broadcasting of bits, i.e., communicating them publicly, does not improve the success rate of the classical protocol.
  - [14] Any dichotomic function,  $\sigma = \pm 1$ , of a two valued variable,  $\xi = \pm 1$  has the form  $\sigma(\xi) = b + c\xi$ , with  $|b|, |c| = 0$  or 1 and  $|b| + |c| = 1$ , i.e., only one term is nonzero.
  - [15] In sequential communication,  $e_k$  depends only on the local variables  $x_k, y_k$  and the message  $e_{k-1}$  from the previous partner, with  $e_k = y_k a_k(x_k) e_{k-1}$ . Recursive expansion of  $e_{k-1}$  results in the same  $e_N$  as above.
  - [16] M. Żukowski, *Phys. Lett. A* **177**, 290 (1993).
  - [17] A. S. Holevo, *Probl. Peredachi Inf.* **9**, 3 (1973) [*Probl. Inf. Transm.* **9**, 177 (1973)].
  - [18] One should not be tempted to exchange the heralded single photons carrying the qubits by bright polarized pulses of light. In such a case, a suitable polarization measurement of the pulses reveals all the encoded input data of any party, i.e. two bits for task A and arbitrarily many for task B. Thus, the communication restriction to  $N - 1$  bits is violated! The attenuation of pulses to single-photon level does not help either. The efficiency of the protocol would be significantly lowered, as the attenuation causes a lot of non-detection events, forcing one to guess the answer most of the time (see description of experiment in main text).
  - [19] Expressing the final results in terms of fidelities, we obtain  $F_{exp,A} = 0.421 \pm 0.010$  for task A, and  $F_{exp,B} = 0.337 \pm 0.006$  for task B. The best classical protocol reaches  $F_{c,A} = 0.25$  for A and  $F_{c,B} \approx 0.164$  for B.
  - [20] C. Schmid *et. al.*, (to be published).